47

USL Report No. 736

AD 635214

Equations of Motion for a Towed Body Moving in a Vertical Plane

KIRK T. PATTON
JEFFREY W. SCHRAM
Engineering Branch
Anti-Submarine Warfare Sonar Division

15 June 1966

Distribution of this document is unlimited.

U. S. Navy Underwater Sound Laboratory
Fort Trumbull, New London, Connecticut

ABSTRACT

Equations of motion are developed for a towed body moving in a plane defined by gravity and the fore-and-aft axis of the ship. Solution of the cross-coupled equations enables predictions to be made of towed body heave, surge, and pitch motions.

The towline is approximated as a spring with one third of its mass added to the body at the towpoint. A discussion of computational techniques to compute masses, damping constants, and spring constants is also presented. Because of the number of terms in the equations for heave, surge, and pitch, the solutions have been programmed in FORTRAN IV.

ADMINISTRATIVE INFORMATION

The analysis described in this report was performed under USL Problem No. 1-650-01-00, Navy Subproject and Task No. SF 101 03 20-8056.

REVIEWED AND APPROVED: 15 June 1966

H. E. Nash
H. E. Nash
Technical Director

R. L. Corkran, Jr.
R. L. Corkran, Jr., Captain, USN
Commanding Officer and Director

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

EQUATIONS OF MOTION FOR A TOWED BODY MOVING IN A VERTICAL PLANE

INTRODUCTION

The designer of a ship-towed system must be able to predict the motions of the towed body. In designing a body that should have minimal response to external excitations, he should have such information available in order to avoid a resonance in any of the body motion modes.

This report presents solutions of the cross-coupled equations of motion for a towed body moving with heave, surge, and pitch motions and for various ship-input conditions. By the use of body and cable constants, one can optimize a design for whichever criterion he must satisfy, be it heave, surge, or pitch, or any combination of the three motions. The analysis is limited to motions in the vertical plane, since additional information that could be derived from solving equations of motion for a body with six degrees of freedom did not appear to be significant at the time. In general, towed bodies have symmetry about a plane defined by the vertical and the fore-and-aft axes.

The primary force input to a cable-towed system is taken to be the vertical force at the point of attachment of the cable on the ship. This limitation on the analysis is reasonable, because forces in this plane will not couple into sway, roll, or yaw for a symmetrical body. The solution may be extended, however, to apply to a body moving with all six degrees of freedom.

Lum[1] solves similar equations by using the Laplace transform technique for the open-loop pitch response of a towed body; however, the results are somewhat difficult to use to find towed-body response for specific input conditions.

---

[1] S. M. Y. Lum, _A Theoretical Investigation of the Body Parameters Affecting the Open-Loop Pitch Response of a Submerged Towed Body_, David Taylor Model Basin Report No. 1369, February 1960 (UNCLASSIFIED).

ANALYSIS

A typical towed system is shown in Fig. 1. The dynamic model of the system is presented in Fig. 2.

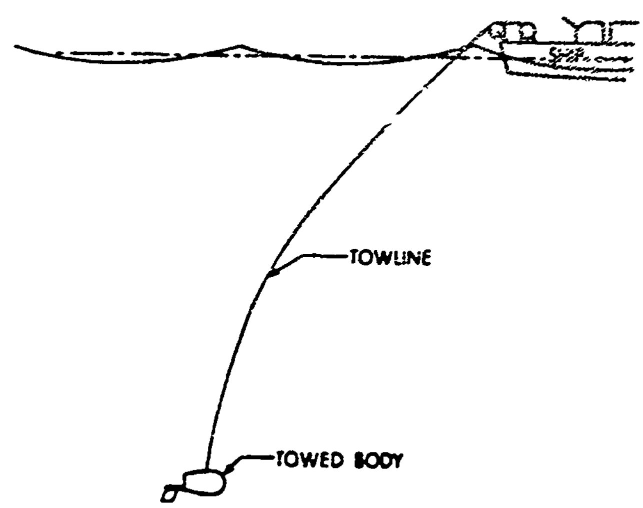

Fig. 1 - A Towed System

The dimensions, masses, spring constants, and damping factors for Fig. 2 are defined as follows:

C_{by} - damping factor of the body alone for motion in the y-direction

C_{bz} - damping factor of the body alone for motion in the z-direction

C_{br} - rotational damping factor of the body alone for motion about the x-axis

C_t - damping factor of the tail alone for motion in the z-direction

D - vertical distance from center of mass to tail center of damping

G - vertical distance from the center of mass to the tail's effective center of resilience and damping.

h - vertical distance from the center of mass to the towpoint

j - vertical distance from the center of mass to the body's center of damping

K_b - spring constant due to change of body lift with angle of attack

K_{cy} - spring constant of deflected towcable in horizontal (y-direction)

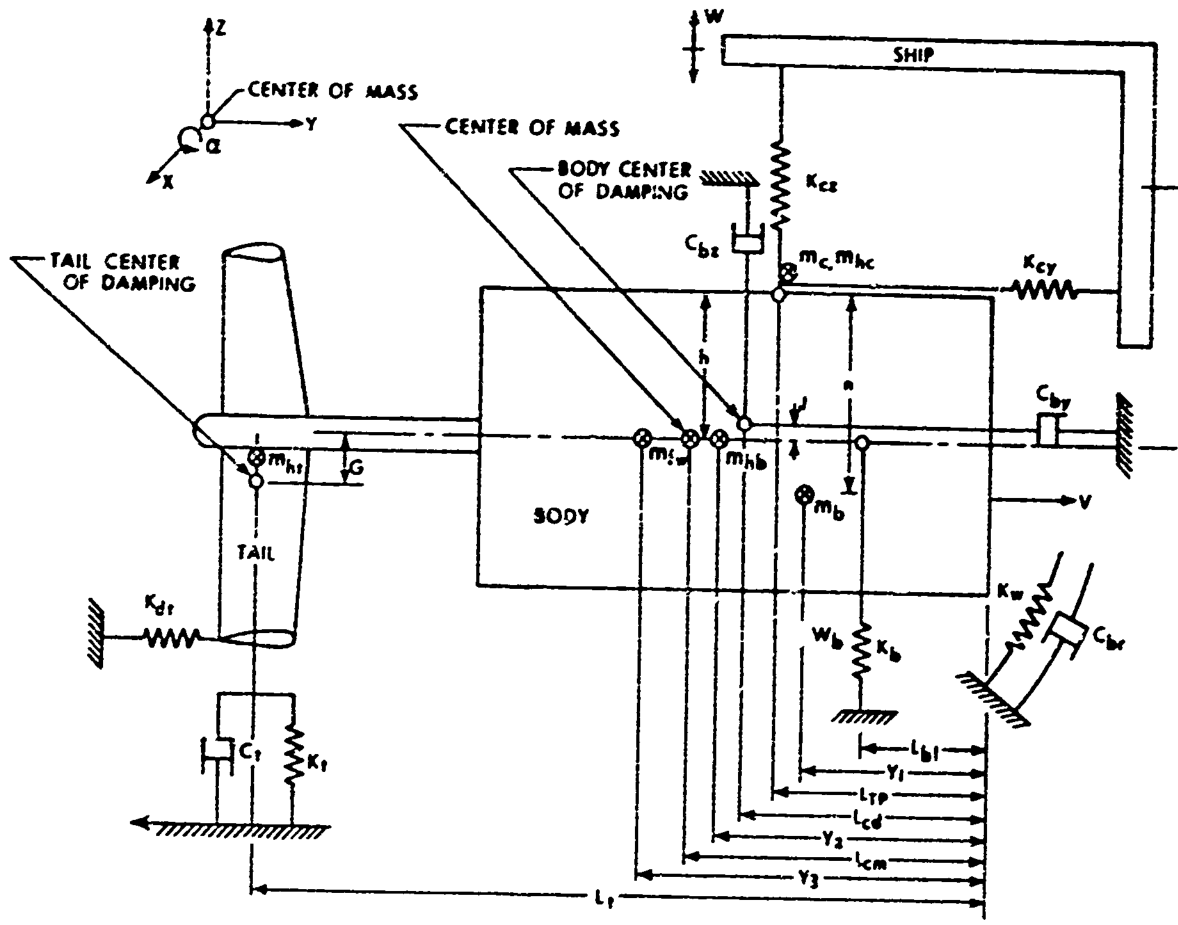

Fig. 2 - Dynamic Model of the Towed System

K_{cz} - spring constant of deflected towcable in vertical (z-direction)

K_t - spring constant due to change of tail lift with angle of attack

K_{dt} - spring constant due to change of tail drag with angle of attack

K_w - rotational spring constant due to pendulous effect of the water weight of the body below the towpoint

L_{bl} - horizontal distance from front of body to the center of body lift

L_{CD} - horizontal distance from front of body to the center of body damping

L_{CM} - horizontal distance from front of body to the center of mass

L_t - horizontal distance from front of body to the effective center of the tail

L_{TP} - horizontal distance from front of body to the towpoint

m_b - mass of the body and tail

m_c - mass of the towcable

m_{fw} - mass of water that floods body

m_{hby} - hydrodynamic mass of the body in the y-direction

m_{hbz} - hydrodynamic mass of the body in the z-direction

m_{hcy} - hydrodynamic mass of the cable in the y-direction

m_{hcz} - hydrodynamic mass of the cable in the z-direction

m_{ht} - hydrodynamic mass of the tail in the z-direction

n - vertical distance from towpoint to center of gravity of the dry body

W - vertical displacement of the towpoint on the ship

w_b - weight of the body in sea water

y_1 - horizontal distance from front of body to center of gravity of the dry body

y_2 - horizontal distance from front of body to center of body hydrodynamic mass

y_3 - horizontal distance from front of body to center of body flood water mass

The center of mass of the body can be located by taking moments about some reference point. For its location in the y-direction

$$L_{CM} = \frac{y_1 m_b + y_2 m_{hbz} + y_3 m_{fw} + L_t m_{ht} + L_{TP}(1/3\, m_c + 1/3\, m_{hcz})}{m_b + m_{hbz} + m_{fw} + m_{ht} + 1/3\, m_c + 1/3\, m_{hcz}}. \tag{1}$$

The mass and hydrodynamic mass terms of the cable are included in Eq. (1), following the usual convention for dynamic problems to include one third of the mass of the spring. A more thorough analysis might be patterned after the work of Dr. L. F. Whicker.[2]

The rotational spring constant K_w is equal to $w_b n$ if angular displacements are assumed to be small. The spring constants K_b and K_t are actually rotational spring constants but are shown as linear springs in the z-direction for ease of computation. The constants can be computed as follows:

[2] L. F. Whicker, "The Oscillatory Motion of Cable-Towed Bodies," Doctoral Thesis, University of California, Berkeley, California, May 1957.

$$K_b = \rho/2\, A_b\, V^2\, (\Delta C_{\ell b}), \tag{2}$$

$$K_t = \rho/2\, A_t\, V^2\, (\Delta C_{\ell t}), \tag{3}$$

and

$$K_{Dt} = \rho/2\, A_t\, V^2\, (\Delta C_{Dt}), \tag{4}$$

where

ρ = fluid density,

A_b = horizontal projected area of the body,

A_t = horizontal projected area of the tail,

$\Delta C_{\ell b}$ = change in lift coefficient of the body with angle of attack,

$\Delta C_{\ell t}$ = change in lift coefficient of the body with angle of attack, and

ΔC_{Dt} = change of drag coefficient of the tail with angle of attack.

A criterion for stability is that the lift moment of the tail must be greater than the lift moment of the body; it can be expressed mathematically by

$$K_t\,(L_t - L_{CM}) > K_b\,(L_{CM} - L_{b\ell}).$$

Using the equations of motion for a resiliently supported body[3] with six degrees of freedom as presented by Harris and Crede,[4]

---

[3]This form of equation of motion is used because of its linear form. Viscous forces are represented as damping constants and inertial history terms are assumed to be negligible.

[4]C. M. Harris and C. E. Crede, Shock and Vibration Handbook, vol. 1, McGraw-Hill Book Company, Inc., New York, 1961.

but restricting the motion to a plane with inputs occurring in that plane only, we can write

$$
\begin{aligned}
M_x = {} & I_{xx}\ddot{\alpha} + \sum (K_{yz}A_y - K_{yy}A_z)\,y_c + \sum (K_{zz}A_y - K_{yz}A_z)(Z_c - W) \\
& + \sum (K_{yy}A_z^2 + K_{zz}A_y^2 - 2K_{yz}A_yA_z)\,\alpha + \sum (C_{yz}A_y' - C_{yy}A_z')\,\dot{y}_c \\
& + \sum (C_{zz}A_y' - C_{yz}A_z')(\dot{z}_c - \dot{W}) + \sum (C_{yy}A_z'^2 + C_{zz}A_y'^2 - 2C_{yz}A_y'A_z')\,\dot{\alpha},
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
F_y = {} & m_y\ddot{y}_c + \sum K_{yy}y_c + \sum K_{yz}(z_c - W) + \sum (K_{yz}A_y - K_{yy}A_z)\,\alpha \\
& + \sum C_{yy}\dot{y}_c + \sum C_{yz}(\dot{z}_c - \dot{W}) + \sum (C_{yz}A_y' - C_{yy}A_z')\,\dot{\alpha},
\end{aligned}
\tag{6}
$$

and

$$
\begin{aligned}
F_z = {} & m_z\ddot{z}_c + \sum K_{yz}y_c + \sum K_{zz}(z_c - W) + \sum (K_{zz}A_y - K_{yz}A_z)\,\alpha \\
& + \sum C_{yz}\dot{y}_c + \sum C_{zz}(\dot{z}_o - \dot{W}) + \sum (C_{zz}A_y' - C_{yz}A_z')\,\dot{\alpha},
\end{aligned}
\tag{7}
$$

where

- A = coordinate distance from the body's center of mass to the elastic center of the resilient element;
- A′ = coordinate distance from the body's center of mass to the effective center of the damping element;
- C = damping factors;
- F = dynamic input force applied at the body's center of mass;
- I = mass movement of inertia;
- K = spring constant;
- m = body mass;
- M = dynamic input moment applied at the body's center of mass;
- W = vertical displacement of the ground point of all vertical springs;

$y, \dot{y}, \ddot{y}$ = displacement, velocity, and acceleration in horizontal direction of the body's center of mass relative to earth coordinates;

$z, \dot{z}, \ddot{z}$ = displacement, velocity, and acceleration in the vertical direction of the body's center of mass relative to earth coordinates;

$\alpha, \dot{\alpha}, \ddot{\alpha}$ = angular displacement, velocity, and acceleration in a vertical plane (defined by y and z) of the body relative to earth coordinates;

and where

subscript c = reference to center of mass;

subscript x = reference to x-direction;

subscript y = reference to y-direction;

subscript z = reference to z-direction; and

subscript α = reference to α-axis of rotation.

The computation of the various coordinates distances A and A′ is as follows:

for the spring representing the cable,

$$A_y = L_{CM} - L_{TP}$$

$$A_z = h ;$$

for the spring due to tail lift,

$$A_y = L_{CM} - L_t$$

A_z = G (negative if effective center of tail lift is below the center of mass);

for the spring due to body lift,

$$A_y = L_{CM} - L_{bl}$$

A_z = O (it is assumed that the effective center of body lift lies on the horizontal plane of the center of mass);

for the center of damping of the body,

$$A'_y = L_{CM} - L_{CD}$$

$A'_z = j$ (negative if the center of damping lies below the center of mass);

for the center of damping of the tail,

$$A'_y = L_{CM} - L_t$$

$A'_z = G$ (negative if the center of tail damping is below the center of mass).

We may now consider the actual towed system and use all masses, elastic elements, damping elements, and input forces in order to apply Eqs. (5), (6), and (7). Rewriting Eqs. (5), (6), and (7) with the above conditions and coordinate distances and assuming that

$K_{yz} = K_{zy} = 0$ for the cable, and

$C_{yz} = C_{zy} = 0$ for the body,

we have

$$\begin{aligned} M_x = {} & (J_b + J_{hb})\ddot{\alpha} + (-K_{ch}\,h)\,y_c + [K_{cv}(L_{CM} - L_{TP})](z_c - W) \\ & + [K_{ch}\,h^2 + K_{cv}(L_{CM} - L_{TP})^2 + K_t(L_t - L_{CM}) + K_b(L_{CM} - L_{b\ell}) + K_w + K_{dt}D]\,\alpha \\ & - (C_{bh}\,j)\dot{y}_c + [C_{bv}(L_{CM} - L_{CD}) - C_t(L_t - L_{CM})]\,\dot{z}_c \\ & + [C_{bh}\,j^2 + C_t(L_t - L_{CM})^2 + C_{bv}(L_{CM} - L_{CD})^2 + C_{br}]\,\dot{\alpha}, \end{aligned} \tag{8}$$

$$F_y = m_{ty}\,\ddot{y}_c + K_{ch}\,y_c - (K_{ch}\,h - K_{dt})\,\alpha + C_{bh}\,\dot{y}_c + [C_{br}/L_b - C_{bh}\,j]\,\dot{\alpha}, \tag{9}$$

and

$$\begin{aligned} F_z = {} & m_{tz}\,\ddot{z}_c + K_{cv}\,(z_c - W) + [K_{cv}\,(L_{CM} - L_{TP}) + K_b - K_t]\,\alpha \\ & + (C_{bv} + C_t)\,\dot{z}_c + [-C_t\,(L_t - L_{CM}) + C_{bv}\,(L_{CM} - L_{CD}) + C_{br}/L_b]\,\dot{\alpha}\ , \end{aligned} \tag{10}$$

where

J_b = mass moment of inertia in the pitch plane about the center of mass;

$J_{h\dot{b}}$ = hydrodynamic mass moment of inertia in the pitch plane about the center of mass;

F_y = forcing function acting on the center of mass in the y-direction;

F_z = forcing function acting on the center of mass in the z-direction;

M_x = forcing function acting about the center of mass in the pitch plane;

m_{ty} = total oscillating mass in the y-direction $(m_{ty} = m_b + m_{hby} + m_{fw} + 1/3\, m_c + 1/3\, m_{hcy})$; and

m_{tz} = total oscillating mass in the z-direction $(m_{tz} = m_b + m_{hbz} + m_{fw} + m_{ht} + 1/3\, m_c + 1/3\, m_{hcz})$.

The spring constant due to body lift is negative in Eq. (8) because its effect is to upset the body if it is displaced. Other assumptions applied to Eqs. (8), (9), and (10) are that cable damping is small compared with body damping and that the center of tail damping lies along the vertical height of the center of mass. It is also assumed that the flood water mass center, the hydrodynamic mass center, and the center of body heave damping lie in the same horizontal plane.

SOLUTION OF EQUATIONS

In order to change the equations of motion (Eqs. (8), (9), and (10)) into dimensionless form, the following steps are taken:

(1) All a terms are multiplied by L_b/L_b, where L_b is the body length.

(2) The two force equations are divided by w_b, the body weight in water.

(3) The moment equation is divided by $w_b L_b$.

Therefore, Eqs. (8), (9), and (10) become

$$
\begin{aligned}
\frac{M_x + K_{cv}(L_{CM} - L_{TP})W}{L_b W_b} = & \left[\frac{J_b + J_{ht}}{L_b^2 W_b}\right](L_b \ddot{a}) - \left[\frac{K_{ch} h}{L_b W_b}\right] y_c + \left[\frac{K_{cv}(L_{CM} - L_{TP})}{L_b W_b}\right] z_c \\
& + \left[\frac{K_{ch} h^2 + K_{cv}(L_{CM} - L_{TP})^2 + K_t(L_t - L_{CM}) + K_b(L_{CM} - L_{bl}) + K_w + K_{dt} D}{L_b^2 W_b}\right](a L_b) \\
& - \left[\frac{C_{bh} j}{L_b W_b}\right] \dot{y}_c + \left[\frac{C_{bv}(L_{CM} - L_{CD}) + C_t(L_{CM} - L_t)}{L_b W_b}\right] \dot{z}_c \\
& + \left[\frac{C_{bh} j^2 + C_t(L_{CM} - L_t)^2 + C_{bv}(L_{CM} - L_{CD})^2 + C_{br}}{L_b^2 W_b}\right](L_b \dot{a}),
\end{aligned}
\tag{I}
$$

$$
\begin{aligned}
\frac{F_y}{W_b} = & \left[\frac{m_{ty}}{W_b}\right] \ddot{y}_c + \left[\frac{K_{ch}}{W_b}\right] y_c - \left[\frac{K_{ch} h - K_{dt}}{W_b L_b}\right] L_b a \\
& + \left[\frac{C_{bh}}{W_b}\right] \dot{y}_c \left[\frac{C_{br}/L_b + C_{bh} j}{W_b L_b}\right] L_b \dot{a}, \text{ and}
\end{aligned}
\tag{II}
$$

$$\frac{F_z + W K_{cv}}{W_b} = \left[\frac{m_{tz}}{W_b}\right]\ddot{z} + \left[\frac{K_{cv}}{W_b}\right] Z_c + \left[\frac{K_{cv}(L_{CM} - L_{TD}) + K_b - K_t}{W_b L_b}\right] L_b \alpha$$

$$+ \left[\frac{C_{bv} + C_t}{W_b}\right]\dot{z} + \left[\frac{C_t(L_t - L_{CM}) - C_{br}(L_{CM} - L_{CD}) + C_{br}}{W_b L_b}\right]\dot{\alpha} L_b . \qquad \text{(III)}$$

For Eqs. (I), (II), and (III), let

$$L_b \alpha = q,$$

$$L_b \dot{\alpha} = \dot{q},$$

and

$$L_b \ddot{\alpha} = \ddot{q}.$$

In the equations that follow, the terms I, II, and III indicate Eqs. (I), (II), and (III), respectively, and the subscript following these terms represents the variable that multiplies the term.

$$I_{\ddot{q}} = \left[\frac{J_b + J_{hb}}{L_b^2 W_b}\right].$$

$$I_y = \left[\frac{K_{ch} h}{L_b W_b}\right].$$

$$I_z = \left[\frac{K_{cv}(L_{CM} - L_{TP})}{L_b W_b}\right].$$

$$I_q = \left[\frac{K_{ch}\,h^2 + K_{cv}\,(L_{CM} - L_{TP})^2 + K_t\,(L_t - L_{CM}) + K_b\,(L_{CM} - L_{b\ell}) + K_w + K_{dt}\,D}{L_b^2\,W_b}\right].$$

$$I_{\dot{y}} = \left[\frac{C_{bh}\,j}{L_b\,W_b}\right].$$

$$I_{\dot{z}} = \left[\frac{C_{bv}\,(L_{CM} - L_{CD}) + C_t\,(L_{CM} - L_t)}{L_b\,W_b}\right].$$

$$I_{\dot{q}} = \left[\frac{C_{bh}\,j^2 + C_t\,(L_{CM} - L_t)^2 + C_{bv}\,(L_{CM} - L_{CD})^2 + C_{br}}{L_b^2\,W_b}\right].$$

$$II_{\ddot{y}} = \left[\frac{m_{ty}}{W_b}\right].$$

$$II_y = \left[\frac{K_{ch}}{W_b}\right].$$

$$II_q = \left[\frac{K_{ch}\,h - K_{dt}}{W_b\,L_b}\right].$$

$$II_{\dot{y}} = \left[\frac{C_{bh}}{W_b}\right].$$

$$II_{\dot{q}} = \left[\frac{C_{bh}\,j}{W_b\,L_b} + \frac{C_{br}}{W_b\,L_b^2}\right].$$

$$III_{\ddot{z}} = \left[\frac{m_{tz}}{W_b}\right].$$

$$III_z = \left[\frac{K_{cv}}{W_b}\right].$$

$$III_q = \left[\frac{K_{cv}(L_{CM} - L_{TP}) + K_b - K_t}{W_b L_b}\right].$$

$$III_{\dot{z}} = \left[\frac{C_{bv} + C_t}{W_b}\right].$$

$$III_{\dot{q}} = \left[\frac{C_{bv}(L_{CM} - L_{CD}) - C_t(L_t - L_{CM}) + C_{br}/L_b}{W_b L_b}\right].$$

$$P = \left[\frac{M_x + K_{cv}(L_{CM} - L_{TP})W}{L_b W_b}\right].$$

$$Y = \left[\frac{F_y}{W_b}\right].$$

$$Z = \left[\frac{F_z + W K_{cv}}{W_b}\right].$$

The equations of motion now become

$$P = I_{\ddot{q}}\ddot{q} - I_y y + I_z z + I_q q - I_{\dot{y}}\dot{y} + I_{\dot{z}}\dot{z} + I_{\dot{q}}\dot{q}, \qquad \text{(IA)}$$

$$Y = II_{\ddot{y}}\ddot{y} + II_y y - II_q q + II_{\dot{y}}\dot{y} - II_{\dot{q}}\dot{q}, \qquad \text{(IIA)}$$

$$Z = III_{\ddot{z}}\ddot{z} + III_z z + III_q q + III_{\dot{z}}\dot{z} + III_{\dot{q}}\dot{q}; \qquad \text{(IIIA)}$$

or in matrix notation,

$$\begin{bmatrix} II_{\ddot{y}} & 0 & 0 \\ 0 & III_{\ddot{z}} & 0 \\ 0 & 0 & I_{\ddot{q}} \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{q} \end{bmatrix} + \begin{bmatrix} II_{\dot{y}} & 0 & -II_{\dot{q}} \\ 0 & III_{\dot{z}} & III_{\dot{q}} \\ -I_{\dot{y}} & I_{\dot{z}} & I_{\dot{q}} \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{q} \end{bmatrix} + \begin{bmatrix} II_y & 0 & -II_q \\ 0 & III_z & III_q \\ -I_y & I_z & I_q \end{bmatrix} \begin{bmatrix} y \\ z \\ q \end{bmatrix} = \begin{bmatrix} Y \\ Z \\ P \end{bmatrix}$$

or

$$M\ddot{Q}(t) + C\dot{Q}(t) + KQ(t) = F(t). \qquad (11)$$

The solution to Eq. (11) involves a complementary and a particular integral. The complementary solution, which is obtained by setting the forcing function $F(t)$ of Eq. (11) equal to zero, is

$$Q_c = e^{-(C/2m)t}(C_1 \cos \omega_{ndt} + C_2 \sin \omega_{ndt}),$$

where

t = time,

C_1 and C_2 = arbitrary constants, and

ω_{nd} = damped natural frequency of the system.

It can be seen that the complementary solution is a transient one that diminishes with time. Therefore, the steady-state condition is described by the particular solution.

To effect the particular solution we let

$$F(t) = F_0 \cos \omega t ,$$

and assume that

$$Q = Q_2 \cos \omega t + Q_1 \sin \omega t,$$

where

ω = forcing frequency.

Taking derivatives and substituting in Eq. (11) we obtain

$$\dot{Q} = -\omega Q_2 \sin \omega t + \omega Q_1 \cos \omega t ,$$

and

$$\ddot{Q} = -\omega^2 Q_2 \cos \omega t - \omega^2 Q_1 \sin \omega t ;$$

therefore, we have

$$\begin{aligned} -\omega^2 M (Q_2 \cos \omega t + Q_1 \sin \omega t) + \omega C (Q_1 \cos \omega t - Q_2 \sin \omega t) \\ + K (Q_2 \cos \omega t + Q_1 \sin \omega t) = F_0 \cos \omega t . \end{aligned} \tag{12}$$

Equation (12) must satisfy the condition that the sum of the forces in the vertical and horizontal directions must equal the inertia forces in those directions. Hence, we have

$$-\omega^2 MQ_2 + \omega CQ_1 + KQ_2 = F, \tag{13}$$

and

$$-\omega^2 MQ_1 - \omega CQ_2 + KQ_1 = 0. \tag{14}$$

Because M, C, and K are matrices, multiplication of these terms is not commutative.

Solving Eq. (14) for Q_2 and substituting into Eq. (13), we obtain

$$Q_2 = \frac{1}{\omega} C^{-1} [K - \omega^2 M] Q_1 ,$$

and

$$\left[\frac{1}{\omega} C^{-1} (K - \omega^2 M) \frac{1}{\omega} C^{-1} (K - \omega^2 M) + 1\right] Q_1 = \frac{1}{\omega} C^{-1} F .$$

Solving for Q_1, we have

$$Q_1 = \left[\frac{1}{\omega} C^{-1} (K - \omega^2 M)^2 + 1\right]^{-1} \frac{1}{\omega} C^{-1} F ,$$

in which

$$C = \begin{bmatrix} II_{\dot{y}} & 0 & -II_{\dot{q}} \\ 0 & III_{\dot{z}} & III_{\dot{q}} \\ -I_{\dot{y}} & I_{\dot{z}} & I_{\dot{q}} \end{bmatrix}$$

and

$$C^{-1} = \begin{bmatrix} C_{yy} & C_{yz} & C_{yq} \\ C_{zy} & C_{zz} & C_{zq} \\ C_{qy} & C_{qz} & C_{qq} \end{bmatrix},$$

where

$$C_{yy} = \mathrm{III}_{\dot z} \mathrm{I}_{\dot q} - \mathrm{III}_{\dot q} \mathrm{I}_{\dot z}/A,$$

$$C_{zz} = \mathrm{I}_{\dot q} \mathrm{II}_{\dot y} - \mathrm{II}_{\dot q} \mathrm{I}_{\dot y}/A,$$

$$C_{zy} = -\mathrm{III}_{\dot q} \mathrm{I}_{\dot y}/A,$$

$$C_{qz} = -\mathrm{II}_{\dot y} \mathrm{I}_{\dot z}/A,$$

$$C_{qy} = \mathrm{I}_{\dot y} \mathrm{III}_{\dot z}/A,$$

$$C_{yq} = \mathrm{II}_{\dot q} \mathrm{III}_{\dot z}/A,$$

$$C_{yz} = -\mathrm{II}_{\dot q} \mathrm{I}_{\dot z}/A,$$

$$C_{zq} = -\mathrm{II}_{\dot y} \mathrm{III}_{\dot q}/A, \quad \text{and}$$

$$C_{qq} = \mathrm{II}_{\dot y} \mathrm{III}_{\dot z}/A,$$

where

$$A = \mathrm{II}_{\dot y} (\mathrm{III}_{\dot z} \mathrm{I}_{\dot q} - \mathrm{III}_{\dot q} \mathrm{I}_{\dot z}) - \mathrm{II}_{\dot q} (\mathrm{I}_{\dot y} \mathrm{III}_{\dot z}) .$$

Performing matrix operations for the terms in the expressions Q_1 and Q_2, we obtain

$$\frac{1}{\omega} C^{-1} (K - \omega^2 M) = \frac{1}{\omega} \begin{bmatrix} C_{yy} & C_{yz} & C_{yq} \\ C_{zy} & C_{zz} & C_{zq} \\ C_{qy} & C_{qz} & C_{qq} \end{bmatrix} \begin{bmatrix} \mathrm{II}_y - \omega^2 \mathrm{II}_{\ddot y} & 0 & -\mathrm{II}_q \\ 0 & \mathrm{III}_z - \omega^2 \mathrm{III}_{\ddot z} & \mathrm{III}_q \\ -\mathrm{I}_y & \mathrm{I}_z & \mathrm{I}_q - \omega^2 \mathrm{I}_{\ddot q} \end{bmatrix},$$

and

$$
\begin{aligned}
\frac{1}{\omega} C^{-1} (K - \omega^2 M) = \frac{1}{\omega} \Big[& C_{yy}(II_y - \omega^2 II_{\ddot{y}}) - I_y C_{yq} \quad C_{yz}(III_z - \omega^2 III_{\ddot{z}}) + C_{yq} I_z \\
& - C_{yz} III_q - C_{yy} II_q + C_{yq}(I_q - \omega^2 I_{\ddot{q}}) \quad C_{zy}(II_y - \omega^2 II_{\ddot{y}}) \\
& - I_y C_{zq} \quad C_{zq} I_z + C_{zz}(III_z - \omega^2 III_{\ddot{z}}) - C_{zy} II_q + C_{zz} III_q \\
& + C_{zq}(I_q - \omega^2 I_{\ddot{q}}) \quad C_{qy}(II_y - \omega^2 II_{\ddot{y}}) - C_{qq} I_y \quad C_{qz}(III_z - \omega^2 III_{\ddot{z}}) \\
& + C_{qq} I_z \quad - C_{qy} II_q + C_{qz} III_q + C_{qq}(I_q - \omega^2 I_{\ddot{q}}) \Big] .
\end{aligned}
$$

Let

$$
\frac{1}{\omega} C^{-1} (K - \omega^2 M) = \frac{1}{\omega} \begin{bmatrix} r_{yy} & r_{yz} & r_{yq} \\ r_{zy} & r_{zz} & r_{zq} \\ r_{qy} & r_{qz} & r_{qq} \end{bmatrix} .
$$

Now forming, we have

$$
\left[\left(\frac{1}{\omega} C^{-1} (K - \omega^2 M) \right)^2 + I \right] = \frac{1}{\omega^2} \begin{bmatrix} r_{yy} & r_{yz} & r_{yq} \\ r_{zy} & r_{zz} & r_{zq} \\ r_{qy} & r_{qz} & r_{qq} \end{bmatrix} \begin{bmatrix} r_{yy} & r_{yz} & r_{yq} \\ r_{zy} & r_{zz} & r_{zq} \\ r_{qy} & r_{qz} & r_{qq} \end{bmatrix} + I
$$

$$
= \begin{bmatrix}
(r_{yy}^2 + r_{yz} r_{zy} + r_{qq} r_{qy})(1/\omega^2) + 1 & (r_{yy} r_{yz} + r_{zz} r_{yz} + r_{qz} r_{yq})(1/\omega^2) & (r_{yy} r_{yq} + r_{yz} r_{zq} + r_{qq} r_{yq})(1/\omega^2) \\
(r_{zy} r_{yy} + r_{zz} r_{zy} + r_{zq} r_{qy})(1/\omega^2) & (r_{zy} r_{yz} + r_{zz}^2 + r_{qz} r_{zq})(1/\omega^2) + 1 & (r_{yq} r_{zy} + r_{zq} r_{zz} + r_{zq} r_{qq})(1/\omega^2) \\
(r_{qy} r_{yy} + r_{qz} r_{zy} + r_{qq} r_{qy})(1/\omega^2) & (r_{yz} r_{qy} + r_{zz} r_{qz} + r_{qq} r_{qz})(1/\omega^2) & (r_{yq} r_{qy} + r_{qq}^2 + r_{qz} r_{zq})(1/\omega^2) + 1
\end{bmatrix}
$$

Let

$$
\left[\left(\frac{1}{\omega} C^{-1} (K - \omega^2 M)\right)^2 + 1\right] = \begin{bmatrix} h_{yy} & h_{yz} & h_{yq} \\ h_{qy} & h_{zz} & h_{zq} \\ h_{qz} & h_{qz} & h_{qq} \end{bmatrix}.
$$

Forming, we have

$$
[(\omega C^{-1} (K - \omega^2 M))^2 + 1]^{-1} = \begin{bmatrix} J_{yy} & J_{yz} & J_{yq} \\ J_{zy} & J_{zz} & J_{zq} \\ J_{qy} & J_{qz} & J_{qq} \end{bmatrix},
$$

where

$$J_{yy} = h_{zz} h_{qq} - h_{zq} h_{qz}/B,$$

$$J_{zy} = h_{zq} h_{qy} - h_{zy} h_{qq}/B,$$

$$J_{qy} = h_{zy} h_{qz} - h_{zz} h_{qy}/B,$$

$$J_{yz} = h_{yq} h_{qz} - h_{yz} h_{qq}/B,$$

$$J_{zz} = h_{yy} h_{qq} - h_{yq} h_{qy}/B,$$

$$J_{qz} = h_{yz} h_{qy} - h_{yy} h_{qz}/B,$$

$$J_{yq} = h_{yz} h_{zq} - h_{zz} h_{yq}/B,$$

$$J_{zq} = h_{yq} h_{zy} - h_{yy} h_{zq}/B, \quad \text{and}$$

$$J_{qq} = h_{yy} h_{zz} - h_{yz} h_{zy}/B,$$

where

$$B = h_{yy} (h_{zz} h_{qq} - h_{zq} h_{qz}) + h_{yz} (h_{zq} h_{qy} - h_{zy} h_{qq}) + h_{yq} (h_{zy} h_{qz} - h_{zz} h_{qy}).$$

Forming, we have

$$\frac{1}{\omega} C^{-1} F = \frac{1}{\omega} \begin{bmatrix} C_{yy} & C_{yz} & C_{yq} \\ C_{zy} & C_{zz} & C_{zq} \\ C_{qy} & C_{qz} & C_{qq} \end{bmatrix} \begin{bmatrix} \bar{Y} \\ \bar{Z} \\ \bar{P} \end{bmatrix}$$

and

$$\frac{1}{\omega} C^{-1} F = \frac{1}{\omega} \begin{bmatrix} C_{yy} \bar{Y} + C_{yz} \bar{Z} + C_{yq} \bar{P} \\ C_{zy} \bar{Y} + C_{zz} \bar{Z} + C_{zq} \bar{P} \\ C_{qy} \bar{Y} + C_{qz} \bar{Z} + C_{qq} \bar{P} \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix} E \\ F \\ G \end{bmatrix} .$$

Finally, we obtain

$$\left[\left(\frac{1}{\omega} C^{-1} (K - \omega^2 M)\right)^2 + I\right]^{-1} \frac{1}{\omega} C^{-1} F = Q_1 = \begin{bmatrix} y_1 \\ z_1 \\ q_1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ z_1 \\ q_1 \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix} J_{yy} & J_{yz} & J_{yq} \\ J_{zy} & J_{zz} & J_{zq} \\ J_{qy} & J_{qz} & J_{qq} \end{bmatrix} \begin{bmatrix} E \\ F \\ G \end{bmatrix}$$

$$Q_1 = \frac{1}{\omega} \begin{bmatrix} J_{yy} E + J_{yz} F + J_{yq} G \\ J_{zy} E + J_{zz} F + J_{zq} G \\ J_{qy} E + J_{qz} F + J_{qq} G \end{bmatrix} = \begin{bmatrix} y_1 \\ z_1 \\ q_1 \end{bmatrix} . \qquad (15)$$

Therefore, we obtain

$$Q_2 = \frac{1}{\omega} \begin{bmatrix} r_{yy} & r_{yz} & r_{yq} \\ r_{zy} & r_{zz} & r_{zq} \\ r_{qy} & r_{qz} & r_{qq} \end{bmatrix} \begin{bmatrix} y_1 \\ z_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} y_2 \\ z_2 \\ q_2 \end{bmatrix} \qquad (16)$$

$$\begin{bmatrix} y_2 \\ z_2 \\ q_2 \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix} r_{yy} y_1 + r_{yz} z_1 + r_{yq} q_1 \\ r_{zy} y_1 + r_{zz} z_1 + r_{zq} q_1 \\ r_{qy} y_1 + r_{qz} z_1 + r_{qq} q_1 \end{bmatrix}$$

and

$$y = y_1 \sin \omega t + y_2 \cos \omega t,$$

$$z = z_1 \sin \omega t + z_2 \cos \omega t, \quad \text{and}$$

$$q = q_1 \sin \omega t + q_2 \cos \omega t.$$

COMMENTS

A simple computer program has been written to effect these complex solutions. This program with necessary inputs is shown in Appendix A.

In working with these equations, one finds that the most difficult task is determining the cable spring constants, hydrodynamic masses, and damping factors since there is very little information in the literature dealing with the computation of these factors for towed bodies. Appendix B discusses the computations.

The solutions have been used to study the motions of the AN/SQA-10 towed body for various ship speeds and sea conditions. The results agree favorably with data taken at Sea States 2 and 5 with a full-size AN/SQA-10 towed body.

SUMMARY

The differential equations of motion have been written for the case of a cable-supported towed body. The solution for a sinusoidal forcing function input is shown in Eqs. (15) and (16). Computation of the inertial terms, spring constants, and damping factors required for Eqs. (12), (13), (14), (15), and (16) enables one to solve for the magnitude and phase angle of motions in surge, heave, and pitch resulting from vertical displacements of the cable end on the ship. A trial-and-error approach can be used to determine the natural frequencies of the system by substituting various frequencies into the solution.

Appendix A

COMPUTER PROGRAM FOR COMPUTATION OF TOWED BODY MOTIONS IN SURGE, HEAVE, AND PITCH FOR A VERTICAL SINUSOIDAL INPUT AT THE SHIP

INPUT TO COMPUTE SURGE, HEAVE, AND PITCH OF A TOWED BODY*

m_b - mass of body and tail

W_{ba} - weight (includes tail) in air

m_{hby} - hydrodynamic mass of body in fore-and-aft direction

m_{hbz} - hydrodynamic mass of body in vertical direction

m_{fw} - flood water mass

m_{ht} - hydrodynamic mass of tail in vertical direction

W_b - body weight (includes tail) in water

y_1 - distance from front of body to body center of gravity in air

y_2 - distance from front of body to body center of vertical hydrodynamic mass

y_4 - distance from front of body to flood water mass center of gravity

L_t - distance from front of body to tail hydrodynamic mass center of gravity and tail damping

L_{TP} - distance from front of body to towpoint

L_{CD} - distance from front of body to body center of damping

L_{bl} - distance from front of body to center of body lift

n - distance from towpoint to body center of gravity in air

n - distance from towpoint to body center of mass in water (positive if above center of mass)

j - distance from center of mass to center of damping (minus if center of damping below center of mass)

---

*Units must be kept homogeneous.

K_{ch} - horizontal cable spring constant

K_{cv} - vertical cable spring constant

K_b - body lift spring constant $\quad C_{\ell b}\, \rho/2\, A_b\, V^2 \quad C_{\ell b} = f(\alpha)$

K_t - tail lift spring constant $\quad C_{\ell t}\, \rho/2\, A_t\, V^2 \quad C_{\ell t} = f(\alpha)$

K_{dt} - tail drag spring constant $\quad C_{dt}\, \rho/2\, A_t\, V^2 \quad C_{dt} = f(\alpha)$

W displacement amplitude of towpoint on ship

ω - angular frequency of ship's pitch motion

J_b - mass moment of inertia of body about center of mass

J_{hb} - hydrodynamic mass moment of inertia of body about center of mass

C_{bv} - vertical damping factor of body alone

C_{bh} - horizontal damping factor of body alone

C_{br} - rotational damping factor of body alone

C_t - vertical damping factor of tail

m_c - mass of towcable

m_{hcz} - vertical hydrodynamic mass of cable

m_{hcy} - horizontal hydrodynamic mass of cable

PRINTOUT

The surge, heave, and pitch will be printed out in the following order.

Real part of surge
Imaginary part of surge
Vector sum of real and imaginary parts of surge
Phase angle between real and imaginary parts of surge
Real part of heave
Imaginary part of heave
Vector sum of real and imaginary parts of heave
Phase angle between real and imaginary parts of heave

Real part of pitch
Imaginary part of pitch
Vector sum of real and imaginary parts of pitch
Phase angle between real and imaginary parts of pitch.

The units of surge and pitch will be commensurate with the units of the input constants. The phase angles are in degrees. The pitch angles are in radians times the body length.

COMPUTER PROGRAM

```
C     EQUATIONS OF MOTION FOR A TOWED BODY          6484            BAKER
$ JOB OCT540004100460
$ IND OCT546060606060
      DIMENSION TTKCH(20),TTKCV(20),TTKB(4),TTKT(4),TOMEG(4),TTMC(5),
     1TTMHCZ(5),TTMHCY(5),IB(20),IS(20),ISS(20),ID(20),ICOM(20),SF(20),
     2 WORD(20),WT(4),TKDT(4),FMT(12)
      READ INPUT TAPE 3,100,TJOB,MOD1
  100 FORMAT(A5,I5)
      IF(JOB - TJOB) 6,2,6
    6 PAUSE 6
    2 WRITE OUTPUT TAPE 4,101,TJOB
  101 FORMAT(1H1,A5//37H EQUATIONS OF MOTION FOR A TOWED BODY//)
      IF(SENSE SWITCH 2) 600,601
  600 READ INPUT TAPE 3,991,MXSEC,MODE,IO
  991 FORMAT(3I6)
      END FILE IO
      READ INPUT TAPE 3,997, (FMT(I),I=1,12)
  997 FORMAT(12A6)
      READ INPUT TAPE3,992,(IB(I),IS(I),ISS(I),ID(I),I=1,5)
  992 FORMAT(12I6)
      READ INPUT TAPE 3,FMT,(SF(I),ICOM(I),I=1,5)
  601 READ INPUT TAPE 3,120,TMB,WBA,TMHBY,TMHBZ,TMFW,WB,Y1,Y2,Y4,TLTP,TL
     1CD,TLBL,TN,TH,TJ,CBV,CBH,CBB,TD
  120 FORMAT(4E15.8)
      READ INPUT TAPE 3,990,INCR,TLB,TMX,FY,FZ
  990 FORMAT(I6,4F9.4)
      READ INPUT TAPE 3,999,NJOB
  999 FORMAT(I4)
      READ INPUT TAPE 3,998,TMHT,TLT,TJB,TJHB,CT
  998 FORMAT(5F10.2)
      DO 200 I = 1,NJOB
      READ INPUT TAPE 3,120,(WT(KK),KK =1,4)
      READ INPUT TAPE 3,120,(TOMEG(KK), KK=1,4)
      READ INPUT TAPE 3,120,(TTMC(J),TTMHCZ(J),TTMHCY(J),J=1,5)
      READ INPUT TAPE 3,120,(TTKCH(J),TTKCV(J), J=1,20)
      READ INPUT TAPE 3, 120,(TTKT(J),TTKB(J),J=1,4)
      READ INPUT TAPE 3,120,(TKDT(J),J=1,4)
      ISUM = 0
      DO 204 KK=1,4
      W = WT(KK)
      OMEGA = TOMEG(KK)
      TKD = TKDT(KK)
      TKT = TTKT(KK)
      TKB = TTKB(KK)
      DO 201 L = 1,5
      TMC = TTMC(L)
      TMHCZ = TTMHCZ(L)
      TMHCY = TTMHCY(L)
      ISUM = ISUM + 1
      TKCH =TTKCH(ISUM)
      TKCV = TTKCV(ISUM)
      TMTY =  TMB + TMHBY +TMFW + TMC/3. + TMHCY/3.
      TMTZ = TMB + TMHBZ  +TMFW + TMHT + TMC/3. +TMHCZ/3.
      TLCM = ( Y1*TMB + Y2*TMHBZ + Y4*TMFW +TLT*TMHT +(TLTP/3.)*(TMC +
     1TMHCZ))/TMTZ
      TKW = WB*TN
      A = TLB*WB
      TIC2= (TJB +TJHB)   /(A*TLB)
```

```
      TIY = (TKCH*TH)/A
      TIZ = (TKCV*(TLCM -TLTP))/A
      TIQ =( TKCH*TH**2 + TKCV*(TLCM - TLTP)**2 + TKT*(TLT - TLCM) +TKD
     1*TD  +TKB*(TLCM - TLBL) + TKW)/(A*TLB)
      TIY1   = (CBH*TJ)/A
      TIZ1   =(CBV*(TLCM -TLCD) + CT*(TLCM - TLT))/A
      TIQ1   =  (CBH*TJ**2 + CBV*(TLCM - TLCD)**2 + CT*(TLCM - TLT)**2 +
     1  CBR )/(A*TLB)
      TIIY2 =  TMTY/WB
      TIIY = TKCH/WB
      TIIQ = (TKCH*TH - TKD)/A
      TIIY1  =  CBH/WB
      TIIQ1 = (CBH*TJ)/A         + CBP/(A*TLB)
      TIIIZ2  = TMTZ/WB
      TIIIZ   = TKCV/WB
      TIIIQ = (TKCV*(TLCM - TLTD) + TKB - TKT)/A
      TIIIZ1 = (CBV +CT)/WB
      TIIIQ1 = ( CBV*(TLCM - TLCD)- CT*(TLT- TLCM) +  CBR/TLB)/A
      PCAP =  ( TMX   + TKCV*(TLCM - TLTP)*W)/A
      YCAP =  FY/WB
      ZCAP = (FZ + W*TKCV)/WB
      IF( MOD1) 701,700,701
  701 AA = TIIY1*( TIIIZ1*TIQ1 - TIIIQ1*TIZ1) - TIIQ1*(TIY1*TIIIZ1)
      CYY =(TIIIZ1*TIQ1- TIIIQ1*TIZ1)/AA
      CZY =(- TIIIQ1*TIY1)/AA
      CQY =(TIY1*TIIIZ1)/AA
      CYZ = (-TIIQ1* TIZ1)/AA
      CZZ =(TIQ1*TIIY1 - TIIQ1*TIY1)/AA
      CQZ =(-TIIY1*TIZ1)/AA
      CYQ = (TIIQ1*TIIIZ1)/AA
      CZQ =(-TIIY1* TIIIQ1)/AA
      CQQ = (TIIY1*TIIIZ1)/AA
      RYY = CYY*(TIIY - OMEGA**2*TIIY2) - TIY*CYQ
      RYZ = CYZ*(TIIIZ - OMEGA**2*TIIIZ2) + CYQ*TIZ
      RYQ = CYZ*TIIIQ - CYY*TIIQ + CYQ*(TIQ -TIQ2*OMEGA**2)
      RZY = CZY*(TIIY - TIIY2*OMEGA**2) - TIY*CZQ
      RZZ = CZQ*TIZ + CZZ*(TIIIZ - TIIIZ2*OMEGA**2)
      RZQ = -CZY*TIIQ + CZZ*TIIIQ + CZQ*(TIQ - TIQ2*OMEGA**2)
      RQY = CQY*(TIIY - TIIY2*OMEGA**2) - CQQ*TIY
      RQZ = CQZ*(TIIIZ -TIIIZ2*OMEGA**2) + CQQ*TIZ
      RQQ = -CQY*TIIQ + CQZ*TIIIQ + CQQ*(TIQ - TIQ2*OMEGA**2)
      HYY = ( RYY**2 + RYZ*RZY +RYQ*RQY)/OMEGA**2   + 1.
      HZY = (RZY*RYY +RZZ*RZY +RZQ*RQY)/OMEGA**2
      HQY = (RQY*RYY +RQZ*RZY +RQQ*RQY)/OMEGA**2
      HYZ = (RYY*RYZ +RZZ*RYZ + RQZ*RYQ)/OMEGA**2
      HZZ = (RZY*RYZ  +RZZ**2 +RQZ*RZQ)/OMEGA**2    + 1.
      HQZ = (RYZ*RQY + RZZ*RQZ +RQQ*RQZ)/OMEGA**2
      HYQ = (RYY*RYQ + RYZ*RZQ + RQQ*RYQ)/OMEGA**2
      HZQ = (RYQ*RZY + RZQ*RZZ + RZQ*RQQ)/OMEGA**2
      HQQ = (RYQ*RQY + RQQ**2 + RQZ*RZQ)/OMEGA**2   + 1.
      BB  = HYY*(HZZ*HQQ -HZQ*HQZ) + HYZ*(HZQ*HQY -HZY*HQQ) + HYQ*(HZY*
     1HQZ - HZZ*HQY)
      TJYY =(HZZ*HQQ -HZQ*HQZ )/BB
      TJZY = (HZQ*HQY - HZY*HQQ)/BB
      TJQY = (HZY*HQZ  -HZZ*HQY)/BB
      TJYZ = (HYQ*HQZ - HYZ*HQQ)/BB
      TJZZ = (HYY*HQQ - HYQ*HQY)/BB
```

COMPUTER PROGRAM (Cont'd)

```
      TJQZ = (HYZ*HQY - HYY*HQZ)/BB
      TJYQ = (HYZ*HZQ - HZZ*HYQ)/BB
      TJZQ = (HYQ*HZY - HYY*HZQ)/BB
      TJQQ = (HYY*HZZ - HYZ*HZY)/BB
      E = CYY*YCAP + CYZ*ZCAP + CYQ*PCAP
      F = CZY*YCAP + CZZ*ZCAP + CZQ*PCAP
      G = CQY*YCAP + CQZ*ZCAP + CQQ*PCAP
      Y1 = (TJYY*E  + TJYZ*F  +TJYQ*G)/OMEGA
      Z1 = (TJZY*E  + TJZZ*F  +TJZQ*G)/OMEGA
      Q1 = (TJQY*E  + TJQZ*F  +TJQQ*G )/OMEGA
      Y2 = (RYY*Y1 + RYZ*Z1 + RYQ*Q1 )/OMEGA
      Z2 = (RZY*Y1 + RZZ*Z1 + RZQ*Q1 )/OMEGA
      Q2  =( RQY*Y1 + RQZ*Z1 + RQQ*Q1)/OMEGA
  710 YAMP = SQRTF( Y1**2 + Y2**2)
      ZAMP = SQRTF(Z1**2 + Z2**2)
      QAMP = SQRTF(Q1**2 + Q2**2)
      IF(SENSE SWITCH 3) 900,901
  900 WRITE OUTPUT TAPE 4,104,TIIY2,TIIIZ2,TIQ2,TIIY1,TIIQ1,TIIIZ1,TIIIQ
     11,TIY1,TIZ1,TIQ1,TIIY,TIIQ,TIIIZ,TIIIQ,TIY,TIZ,TIQ,YCAP,ZCAP,PCAP
  104 FORMAT(1H , 8H TIIY2 =,F15.8, 9H TIIIZ2 =,F15.8,7H TIQ2 =,F15.8,
     18H TIIY1 =,F15.8, 8H TIIQ1 =,F15.8,/ 9H TIIIZ1 =,F15.8, 9H TIIIQ1
     2=,F15.8,7H TIY1 =,F15.8,7H TIZ1 =,F15.8,7H TIQ1 =,F15.8,/ 7H TIIY
     3=,F15.8,7H TIIQ =,F15.8,8H TIIIZ =,F15.8,8H TIIIQ =,F15.8,6H TIY =
     4,F15.8/6H TIZ =,F15.8,6H TIQ =,F15.8,7H YCAP =,F15.8,7H ZCAP =,F15.8, 7
     5,8,7H PCAP =,F15.8///)
  901 WRITE OUTPUT TAPE 4,151
  151 FORMAT(1H0,117H    MHT          LT        CT     JB        JHB      OMEGA
     1 KT        KB          MC       MHCZ      MHCY        KCH         KCV            W
     2)
      WRITE OUTPUT TAPE 4,150,TMHT,TLT,CT,TJB,TJHB,OMEGA,TKT,TKB,TMC,TMH
     1CZ,TMHCY,TKCH,TKCV,W
  150 FORMAT(1H0,11F8.2,2F10.2,F8.2)
      IF(MOD1) 703,704,703
  704 CALL ACTN(Y1,Y2,THETA,I)
      YANGL = THETA
      CALL ACTN(Z1,Z2,THETA,I)
      ZANGL = THETA
      CALL ACTN(Q1,Q2,THETA,I)
      QANGL = THETA
      Y3 = -Y1*COSF(YANGL) - Y2*SINF(YANGL)
      Z3 = -Z1*COSF(ZANGL) - Z2*SINF(ZANGL)
      Q3 = -Q1*COSF(QANGL) - Q2*SINF(QANGL)
      GO TO 705
  703 CALL ACTN(Y2,Y1,THETA,I)
      YANGL = THETA
      CALL ACTN(Z2,Z1,THETA,I)
      ZANGL = THETA
      CALL ACTN (Q2,Q1,THETA,I)
      QANGL = THETA
      Y3 = -Y1*SINF(YANGL) - Y2*COSF(YANGL)
      Z3 =-Z1*SINF(ZANGL) - Z2*COSF(ZANGL)
      Q3 = -Q1*SINF(QANGL) - Q2*COSF(QANGL)
  705 IF( Y3) 606,607,607
  606 IF(Z3)      608,607,607
  608 IF(Q3) 609,607,607
  607 WRITE OUTPUT TAPE 4,610, Y3,Z3,Q3
  610 FORMAT(1H , 18H ERROR NOT MAXIMUM, 3F20.8)
  609 WRITE OUTPUT TAPE 4,156,Y1,Y2,Z1,Z2,Q1,Q2,YAMP,ZAMP,QAMP,YANGL,ZAN
     1GL,QANGL
```

COMPUTER PROGRAM (Cont'd)

```
156 FORMAT(1H0, 5H Y1 =,F15.8, 5H Y2 =,F15.8,5H Z1 =,F15.8,5H Z2 =,F15.8
   1,8, 5H Q1 =,F15.8, 5H Q2 =,F15.8//7H YAMP =,F15.8,7H ZAMP =,F15.8,
   27H QAMP =,F15.8//8H YANGL =,F15.8,8H ZANGL =,F15.8,8H QANGL =,F15.
   38)
    IF(SENSE SWITCH 1) 620,621
620 YID = -TIY
    CONST = 0
    YD1 = -TIY1
    QIID1 = -TIIQ1
    QIID = -TIIQ
603 FORMAT(3F20.8)
    WRITE OUTPUT TAPE 6,603,TIQ2,TIQ1,TIQ,CONST,YD1,YID,CONST,TIZ1,TIZ
   1,CONST,PCAP,CONST
    WRITE OUTPUT TAPE 6,603,CONST,QIID1,QIIQ,TIIY2,TIIY1,TIIY,CONST,CO
   1NST,CONST,CONST,YCAP,CONST
    WRITE OUTPUT TAPE 6,603,CONST,TIIIQ1,TIIIQ,CONST,CONST,CONST,TIIIZ
   12,TIIIZ1,TIIIZ,CONST,ZCAP,CONST
621 IF(SENSE SWITCH 2)602,201
602 CALL SETUP( IB(1),5,IS(1),ID(1),ISS(1),IO)
    TINC = INCR
    TINCR = 1.0/TINC
    TMAX = 6.281853/OMEGA
    IJ = TMAX/TINCR + 1.
    DO 995 JJ = 1,IJ
    TI = JJ - 1
    ANG = TI*TINCR*OMEGA
    Y = Y1*COSF(ANG) + Y2*SINF(ANG)
    Z = Z1*COSF(ANG) + Z2*SINF(ANG)
    Q = Q1*COSF(ANG) + Q2*SINF(ANG)
    X = COSF(ANG)
    WORD(1) = ANG
    WORD(2) = Y
    WORD(3) = Z
    WORD(4) = Q
    WORD(5) = X
    IF(SENSE SWITCH 3) 500,501
500 WRITE OUTPUT TAPE 4,502, (WORD(LL),LL=1,5)
502 FORMAT(1H ,5F15.8)
501 CALL PLOT(WORD(1),SF(1),NXSEC,MODE,ICOM(1),IO)
995 CONTINUE
    CALL CLOSE(IO,5)
201 CONTINUE
204 CONTINUE
200 CONTINUE
    IF (SENSE SWITCH 1)622,623
622 END FILE 6
    END FILE 6
623 READ INPUT TAPE 3, 109, ISTAR
109 FORMAT(A1)
    IF(IND - ISTAR) 55,5, 55
 55 PRINT 110, IJOB
110 FORMAT(1H1,A5//30H,SHOULD BE END CARD BUT IS NOT)
    END FILE 4
    STOP 555
  5 END FILE 4
    STOP 5
    END(1,1,0,1,1)
```

Appendix B

COMPUTATION OF HYDRODYNAMIC MASS, DAMPING, AND SPRING CONSTANTS

COMPUTATION OF HYDRODYNAMIC MASS CONSTANTS

As shown in Eq. (11), the mass matrix

$$\begin{bmatrix} m_{ty} & 0 & 0 \\ 0 & m_{tz} & 0 \\ 0 & 0 & J_b + J_{hb} \end{bmatrix}$$

must be evaluated. If the body has a plane of symmetry defined by gravity and the fore-and-aft direction, and if the coordinate axes are along the principal axes of inertia, this matrix in its complete form would be

$$\begin{bmatrix} m_{tyy} & 0 & m_{ty\alpha} \\ 0 & m_{tzz} & m_{tz\alpha} \\ m_{t\alpha y} & m_{t\alpha z} & m_{t\alpha\alpha} \end{bmatrix} ,$$

where

m_t is the total mass in a given direction.

The total mass is composed of the body mass (or rotational inertia), the flood-water mass (or rotational inertia), and the hydrodynamic mass (or rotational inertia). The cross-coupling "masses," $m_{ty\alpha}$;

m_{tzx} , $m_{\alpha y}$ and m_{tz} , are hydrodynamic masses only, because of the above choice of coordinate axes.

Hydrodynamic masses for bodies of various shapes moving in translation have been determined experimentally by Patton.[B1] The effects of frequency and displacement amplitude have been investigated by Miller.[B2] Hydrodynamic mass moments of inertia can be computed for ellipsoids in accordance with the methods of Zahm.[B3] Information can not be found in the literature that enables one to compute the cross-coupled hydrodynamic masses. Consequently, the mass matrix used in Eq. (12) has all off-diagonal elements equal to zero.

COMPUTATION OF DAMPING CONSTANTS

It is extremely difficult to find data in the literature that enable one to compute the damping constants for a towed body. Newman[B4] presents an analytical technique to compute damping coefficients of ellipsoids. Patton[B5] presents experimental data for model towed bodies moving in translational motion. These data can be scaled to a full-size body if scaling laws for size and frequency can be determined. This method was used in the computation of damping constants for the AN/SQA-11 VDS towed body.[B6] It was assumed that the damping varies with the cube of the body size.

[B1] K. T. Patton, "An Experimental Determination of Hydrodynamic Masses and Mechanical Impedances," Master of Science Thesis, University of Rhode Island, Kingston, Rhode Island, May 1965. (Also published as USL Report No. 677, 5 October 1965).

[B2] R. J. Miller, "The Effect of Frequency and Oscillatory Amplitude on Hydrodynamic Mass," Master of Science Thesis, University of Rhode Island, Kingston, Rhode Island, May 1965.

[B3] A. F. Zahm, Flow and Force Equations for a Body Revolving in a Fluid, National Advisory Committee for Aeronautics Report No. 323, 1929.

[B4] J. N. Newman, "The Damping of an Oscillating Ellipsoid Near a Free Surface," Journal of Ship Research, vol. 5, no. 3, December 1961, pp. 44-58.

[B5] K. T. Patton, op. cit. (see footnote B1 above).

[B6] K. T. Patton, "Preliminary Towing Tests of the AN/SQA-11 Dense Body Model," USL Technical Memorandum No. 933-057-65, 17 March 1965 (CONFIDENTIAL).

Experimentally measured rotational damping constants and cross-coupled damping constants can not be found in the literature. The damping of the AN/SQA-11 towed body oscillating in the pitch mode was approximated by computing the damping on an equivalent ellipsoid after the method of Newman.

COMPUTATION OF SPRING CONSTANTS

The spring constants due to the body lift and the tail lift have been considered in the text. A certain amount of judgment is called for at this point because the springs may be nonlinear. (ΔC_{ℓ_b} and ΔC_{ℓ_t} may be nonlinear.) One must estimate the pitch amplitude and substitute the best approximate linear values for the actual nonlinear values.

The cable spring constants K_{ch} and K_{cv} can be computed by considering two springs in series. The first spring represents the axial extension of the cable under some load. This spring constant is given by

$$K_1 = \frac{AE}{L},$$

where

A = cross-sectional area of the cable,

E = cable modulus of elasticity, and

L = length of the cable.

The second spring in the series combination is due to the changes in configuration of the towcable because of changes in load at the towed body. First, the equilibrium configuration of the system is computed by using suitable tables or, possibly, the computer

solution programmed by Cuthill.[B7] The drag of the body is then increased by some amount typical of the dynamic loads existing in the cable. The new configuration is computed, and the change in trail distance is used to compute a spring constant K_{yy}. The change in depth can be used to compute a cross-coupled spring constant K_{yz}. The same process is undertaken to compute the spring constants K_{zz} and K_{zy} by increasing the weight of the body.

Equivalent spring constants K_{ch} and K_{cv} are computed as follows:

$$K_{ch} = \frac{K_{yy} K_1 \sin \phi_0}{K_{yy} + K_1 \sin \phi_0},$$

and

$$K_{cv} = \frac{K_{zz} K_1 \cos \phi_0}{K_{zz} + K_1 \cos \phi_0},$$

where

ϕ_0 is the angle (from the vertical) of the towline at the towed body.

The spring constants K_{ch} and K_{cv} are not constant because the springs that they represent are nonlinear. One should compute a number of deflections for different increments of drag and weight to approximate the nonlinear springs in the yy- and zz-directions with linear spring constants K_{ch} and K_{cv}.

[B7] E. Cuthill, A FORTRAN Program for the Calculation of the Equilibrium Configuration of a Flexible Cable in a Uniform Stream, David Taylor Model Basin Report No. 182, May 1963 (UNCLASSIFIED).

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) U. S. Navy Underwater Sound Laboratory Fort Trumbull, New London, Connecticut	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED
	2b. GROUP

3. REPORT TITLE
EQUATIONS OF MOTION FOR A TOWED BODY MOVING IN A VERTICAL PLANE

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Research Report

5. AUTHOR(S) (Last name, first name, initial)
Patton, Kirk T.
Schram, Jeffrey W.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
15 June 1966	40	10

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. SF 101 03 20 Task 8056	736
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	U. S. Navy

13. ABSTRACT

Equations of motion are developed for a towed body moving in a plane defined by gravity and the fore-and-aft axis of the ship. Solution of the cross-coupled equations enables predictions to be made of towed body heave, surge, and pitch motions.

The towline is approximated as a spring with one third of its mass added to the body at the towpoint. A discussion of computational techniques to compute masses, damping constants, and spring constants is also presented. Because of the number of terms in the equations for heave, surge, and pitch, the solutions have been programmed in FORTRAN IV.

DD FORM 1473

UNCLASSIFIED
Security Classification

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Equations of Motion						
Ship-Towed Systems						
Cross-Coupled Equations of Motion						
Ship-Towed Bodies						
Variable Depth Sonar						
VDS						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (*corporate author*) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (*either by the originator or by the sponsor*), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (*paying for*) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.